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Hierarchy of piecewise non-linear maps with non-ergodic behaviour

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Abstract

We study in detail the dynamics of a hierarchy of piecewise maps generated by a one-parameter family of trigonometric chaotic maps and a one-parameter family of elliptic chaotic maps of the cn and sn type. We calculate the Lyapunov exponent and Kolmogorov–Sinai entropy of the these maps with respect to a control parameter. Non-ergodicity of these piecewise maps is proven analytically and investigated numerically. The invariant measure of these maps, which are not equal to 1 or 0, appears to be characteristic for non-ergodic behaviour. A quantity of interest is the Kolmogorov–Sinai entropy, which, for these maps, is smaller compared with the sum of the positive Lyapunov exponents, and it confirms the non-ergodicity of the maps.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction

Ergodic theory is a branch of dynamic systems dealing with questions of average. Sometimes—even for chaotic systems—ergodic theory can make long-term predictions about the average behaviour, starting from the initial data of limited accuracy. The ergodic theory of chaos has been studied in detail by Eckmann and Ruelle [1, 2] and others, whereas non-ergodic mathematical models scarcely exist. It is shown that the simplest one-dimensional dynamic systems satisfying the indecomposability assumption (and even the assumption of topological transitivity) may be non-ergodic, which shows that restrictions of this type are not quite reasonable in the context of general dynamical systems [3]. Furthermore, ergodicity has not been proven for some systems and, sometimes, we have to face the problem of non-ergodicity of many chemical and physical systems, especially in solid systems (e.g. see [4–7]). It is shown that candidates within solid-state systems with extremely slow dynamics often have impurities remaining from their synthesis; hence, it may always be suspected that non-ergodicity is due to disorder [8]. Furthermore, there are specific examples of arbitrarily small perturbations to ergodic systems, which then behave non-ergodically [9–11]. Studies of the origins of non-ergodicity and slow dynamics of polymer gels [12] and effects of temperature and swelling on chain dynamics in sol–gel transition [13, 14] and also non-ergodic transition in colloidal gelation [15] are some examples of research activity dealing with non-ergodicity.

The aim of the present paper is twofold: to introduce the piecewise maps with their invariant measure and to clarify their non-ergodic behaviour. We present the hierarchies of piecewise maps generated by a one-parameter family of chaotic maps and one-parameter families of elliptic chaotic maps of cn and sn types. In particular, we argue that these maps satisfy the non-ergodic assumption. In our definition of a piecewise map, we assume that it consists of components that can describe fixed-point and chaotic behaviour in accordance with various values of the parameter of the map. The paper is organized as follows. Section 2 gives the definitions of piecewise non-linear maps with complete boundary conditions associated with the one-parameter families of chaotic maps (section 2.1) and one-parameter families of elliptic chaotic maps of cn and sn types (section 2.2). In section 3, we discuss the invariant measures of the piecewise maps and, in section 4, the Kolmogorov–Sinai (KS) entropy and Lyapunov exponent (LE) of the piecewise maps are studied. In section 5, we review the ergodic theory and, in section 6, we give our results on the KS entropy and LE of the piecewise maps and we shall explain how they behave non-ergodic. In addition, there is a concluding section and two appendices.

2. Piecewise non-linear maps

We first review the one-parameter families of trigonometric chaotic maps and one-parameter families of elliptic chaotic maps of cn and sn types, which are used to construct the piecewise map. The one-parameter chaotic maps [16] are defined as the ratio of polynomials of degree N:

$$\phi_N^{(1)}(x,\alpha) = \frac{\alpha^2(1+(-1)^N {}_2F_1(-N,N,\frac{1}{2},x))}{(\alpha^2+1)+(\alpha^2-1)(-1)^N {}_2F_1(-N,N,\frac{1}{2},x)} = \frac{\alpha^2(T_N(\sqrt{x}))^2}{1+(\alpha^2-1)(T_N(\sqrt{x}))^2} \tag{2.1}$$

$$\begin{aligned}\phi_N^{(2)}(x,\alpha) &= \frac{\alpha^2(1-(-1)^N {}_2F_1(-N,N,\frac{1}{2},(1-x)))}{(\alpha^2+1)-(\alpha^2-1)(-1)^N {}_2F_1(-N,N,\frac{1}{2},(1-x))} \\ &= \frac{\alpha^2(U_N(\sqrt{(1-x)}))^2}{1+(\alpha^2-1)(U_N(\sqrt{(1-x)}))^2},\end{aligned} \tag{2.2}$$

where N is an integer greater than 1. Also,

$${}_2F_1(-N,N,\tfrac{1}{2},x) = (-1)^N\cos(2N\arccos\sqrt{x}) = (-1)^N T_{2N}(\sqrt{x})$$

is the hypergeometric polynomials of degree N, and $T_N(x)$ and $U_N(x)$ are Chebyshev polynomials of types I and II, respectively. The conjugate maps of the one-parameter families of chaotic maps, which are used to derive their invariant measure and to calculate their KS entropy, are defined as

$$\tilde{\phi}_N^{(1)}(x,\alpha) = h \circ \phi_N^{(1)}(x,\alpha) \circ h^{-1} = \frac{1}{\alpha^2}\tan^2(N\arctan\sqrt{x}), \tag{2.3}$$

$$\tilde{\phi}_N^{(2)}(x,\alpha) = h \circ \phi_N^{(2)}(x,\alpha) \circ h^{-1} = \frac{1}{\alpha^2}\cot^2\left(N\arctan\frac{1}{\sqrt{x}}\right). \tag{2.4}$$

Conjugacy means that the invertible map $h(x) = (1-x)/x$ maps $I = [0, 1]$ into $[0, \infty)$. Here, we present examples of these types, which have been considered in the present paper:

$$\phi_2^{(1)} = \frac{\alpha^2(2x-1)^2}{4x(1-x)+\alpha^2(2x-1)^2},$$

$$\phi_2^{(2)} = \frac{4\alpha^2 x(1-x)}{1+4(\alpha^2-1)x(1-x)},$$

$$\phi_3^{(1)} = \phi_3^{(2)} = \frac{\alpha^2 x(4x-3)^2}{\alpha^2 x(4x-3)^2+(1-x)(4x-1)^2}.$$

Now, we review a hierarchy of one-parameter families of elliptic cn and sn types, which have been used to construct the piecewise maps with non-ergodic behaviour. These kinds of maps are defined as the ratios of Jacobian elliptic functions of cn and sn types through the following equation [17]:

$$\phi_N^{(1)}(x,\alpha) = \frac{\alpha^2(\mathrm{cn}(N\mathrm{cn}^{-1}(\sqrt{x})))^2}{1+(\alpha^2-1)(\mathrm{cn}(N\mathrm{cn}^{-1}(\sqrt{x})))^2}, \tag{2.5}$$

$$\phi_N^{(2)}(x,\alpha) = \frac{\alpha^2(\mathrm{sn}(N\mathrm{sn}^{-1}(\sqrt{x})))^2}{1+(\alpha^2-1)(\mathrm{sn}(N\mathrm{sn}^{-1}(\sqrt{x})))^2}, \tag{2.6}$$

where α is the control parameter. For $N = 2$, we have

$$\phi_2^{(1)}(x,\alpha) = \frac{\alpha^2((1-k^2)(2x-1)+k^2x^2)^2}{(1-k^2+2k^2x-k^2x^2)^2+(\alpha^2-1)((1-k^2)(2x-1)+k^2x^2)^2}$$

$$\phi_2^{(2)}(x,\alpha) = \frac{4\alpha^2 x(1-k^2x)(1-x)}{(1-k^2x^2)^2+4x(1-x)(\alpha^2-1)(1-k^2x)}.$$

It has been proven [17] that, for small values of the parameter K of the elliptic function, these maps are topologically conjugate to the one-parameter families of chaotic maps.

2.1. Piecewise maps generated by one-parameter families of trigonometric chaotic maps

Now, we introduce a hierarchy of piecewise maps generated by one-parameter families of trigonometric chaotic maps given by equations (2.1) and (2.2). To define a piecewise map constructed from a one-parameter chaotic map, we need to take into account boundary

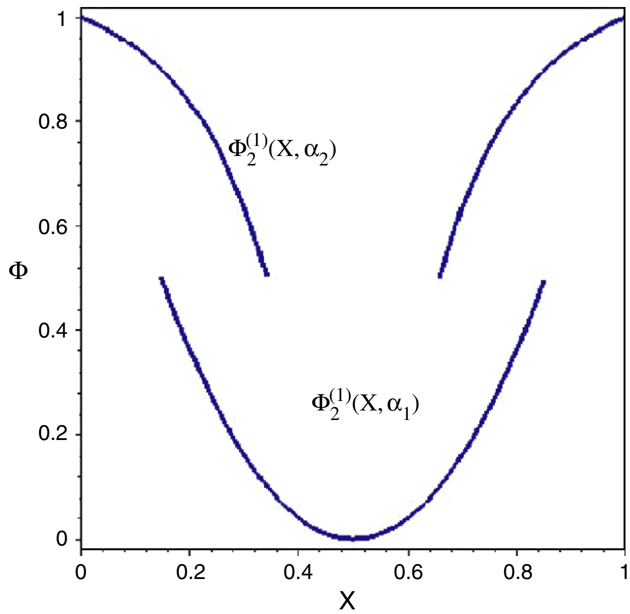

Figure 1. Plot of $\phi_2^{(1)}(x, \alpha)$ for $\alpha_1 = 3$ and $\alpha_2 = 1$.

conditions; namely, we have to choose the states on the phase space. For the piecewise map $\phi_N^{(1)}$ with even N, we have

$$\phi_N^{(1)}(x, \alpha) = \begin{cases} \phi_N^{(1)}(x, \alpha_1), & \alpha_1 \in [N, \infty), \\ \phi_N^{(1)}(x, \alpha_2), & \alpha_2 \in [0, N]. \end{cases}$$

The ranges of the parameters α_1 and α_2 in the maps $\phi_N^{(1)}(x, \alpha_1)$ and $\phi_N^{(1)}(x, \alpha_2)$ are chosen to guarantee, respectively, chaotic behaviour and two fixed points at $x = 0$ and 1. Figure 1 shows the plot of $\phi_2^{(1)}(x, \alpha)$ for $\alpha_1 = 3$ and $\alpha_2 = 1$. In $\phi_N^{(1)}(x, \alpha_1)$ and $\phi_N^{(1)}(x, \alpha_2)$, x is limited to $\acute{x} \in [0.152, 0.848]$ and $\ddot{x} \in [0, 0.352] \cup [0.647, 1]$ respectively. For a given $y_0 = 0.5$, $\acute{x}$ are the roots of $\phi_N^{(1)}(x, \alpha_1) = y_0$ and, similarly, $\ddot{x}$ are the roots of $\phi_N^{(1)}(x, \alpha_2) = y_0$. For the piecewise map $\phi_N^{(2)}(x, \alpha)$ with even N, we have

$$\phi_N^{(2)}(x, \alpha) = \begin{cases} \phi_N^{(2)}(x, \alpha_1), & \alpha_1 \in [0, 1/N], \\ \phi_N^{(2)}(x, \alpha_2), & \alpha_2 \in [1/N, \infty). \end{cases}$$

The ranges of the parameters α_1 and α_2 in the maps $\phi_N^{(2)}(x, \alpha_1)$ and $\phi_N^{(2)}(x, \alpha_2)$ are chosen to guarantee, respectively, two fixed points at $x = 0$ and 1 and chaotic behaviour. Figure 2 shows the plot of $\phi_2^{(2)}(x, \alpha)$ for $\alpha_1 = 0.25$ and $\alpha_2 = 0.75$. In $\phi_N^{(2)}(x, \alpha_1)$ and $\phi_N^{(2)}(x, \alpha_2)$, x is limited to $\acute{x} \in [0, 0.378] \cup [0.621, 1]$ and $\ddot{x} \in [0.2, 0.8]$ respectively. For a given $y_0 = 0$, $\acute{x}$ are the roots of $\phi_N^{(2)}(x, \alpha_1) = y_0$ and, similarly, $\ddot{x}$ are the roots of $\phi_N^{(2)}(x, \alpha_2) = y_0$. For the piecewise map $\phi_N^{(1,2)}$ with odd N, we have

$$\phi_N^{(1,2)}(x, \alpha) = \begin{cases} \phi_N^{(1,2)}(x, \alpha_1), & \alpha_1 \in [1/N, N], \\ \phi_N^{(1,2)}(x, \alpha_2), & \alpha_2 \in [0, 1/N] \cup [N, \infty). \end{cases}$$

This map has chaotic behaviour for α_2 and has a fixed point in $x = 0$ for α_1. Figure 3 shows the plot of $\phi_3^{(1,2)}(x, \alpha)$ for $\alpha_1 = 1.5$ and $\alpha_2 = 0.2$. In $\phi_N^{(1,2)}(x, \alpha_1)$ and $\phi_N^{(1,2)}(x, \alpha_2)$, $\mathbf{x}$ is limited to

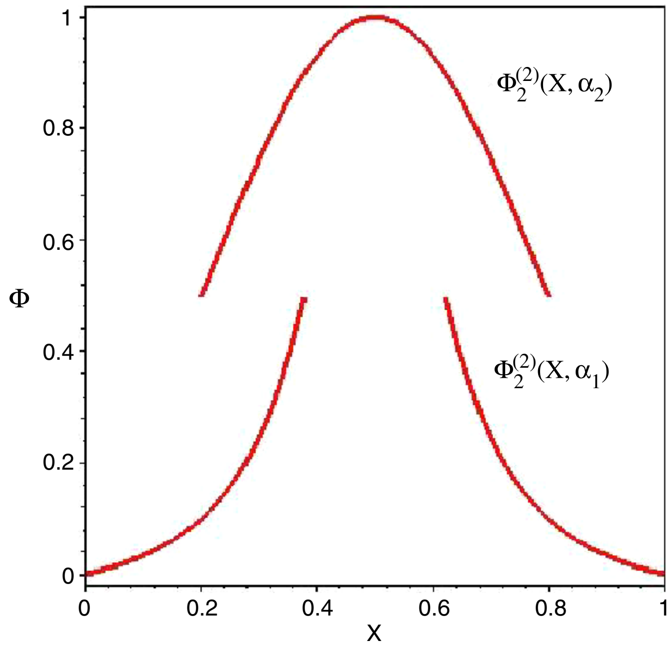

Figure 2. Plot of $\phi_2^{(2)}(x, \alpha)$ for $\alpha_1 = 0.25$ and $\alpha_2 = 0.75$.

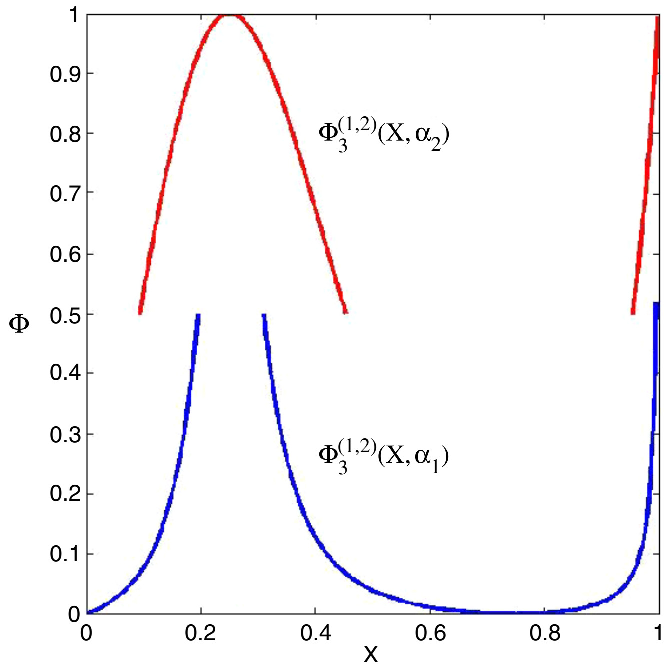

Figure 3. Plot of $\phi_3^{(1,2)}(x, \alpha)$ for $\alpha_1 = 1.5$ and $\alpha_2 = 0.2$.

$\acute{x} \in [0, 0.2] \cup [0.315, 1]$ and $\ddot{x} \in [0.086, 0.453] \cup [0.947, 1]$, respectively. For a given $y_0 = 0.5$, $\acute{x}_i$ are the roots of $\phi_N^{(2)}(x, \alpha_1) = y_0$ and, similarly, $\ddot{x}_i$ are the roots of $\phi_N^{(2)}(x, \alpha_2) = y_0$.

2.2. *One-parameter families of elliptic chaotic maps of cn and sn types*

Here we first review a hierarchy of one-parameter families of elliptic cn and sn types that have been used to construct the piecewise maps with non-ergodic behaviour. These kinds of maps are defined as the ratio of Jacobian elliptic functions of the cn and sn type through the following equation [17]:

$$\phi_N^{(1)}(x, \alpha) = \frac{\alpha^2 (\mathrm{cn}(N \mathrm{cn}^{-1}(\sqrt{x})))^2}{1 + (\alpha^2 - 1)(\mathrm{cn}(N \mathrm{cn}^{-1}(\sqrt{x})))^2},$$

$$\phi_N^{(2)}(x, \alpha) = \frac{\alpha^2 (\mathrm{sn}(N \mathrm{sn}^{-1}(\sqrt{x})))^2}{1 + (\alpha^2 - 1)(\mathrm{sn}(N \mathrm{sn}^{-1}(\sqrt{x})))^2},$$

where α is a control parameter. For $N = 2$, we have

$$\phi_2^{(1)}(x, \alpha) = \frac{\alpha^2((1 - k^2)(2x - 1) + k^2 x^2)^2}{(1 - k^2 + 2k^2 x - k^2 x^2)^2 + (\alpha^2 - 1)((1 - k^2)(2x - 1) + k^2 x^2)^2}$$

$$\phi_2^{(2)}(x, \alpha) = \frac{4\alpha^2 x(1 - k^2 x)(1 - x)}{(1 - k^2 x^2)^2 + 4x(1 - x)(\alpha^2 - 1)(1 - k^2 x)}.$$

It has been proven [17] that, for small values of the parameter K of the elliptic function, these maps are topologically conjugate to the one-parameter family of chaotic maps. Similar to the ones introduced in the previous section, piecewise elliptic maps can be introduced. As an example for a piecewise elliptic map $\phi_2^{(2)}(x, \alpha)$, we have

$$\phi_2^{(2)}(x, \alpha) = \begin{cases} \phi_2^{(2)}(x, \alpha_1), & \alpha_1 \in [0, 1/N], \\ \phi_2^{(2)}(x, \alpha_2), & \alpha_2 \in [1/N, \infty). \end{cases}$$

The range of parameters α_1 and α_2 in the maps $\phi_2^{(2)}(x, \alpha_1)$ and $\phi_2^{(2)}(x, \alpha_2)$ are chosen to guarantee, at the same time, two fixed points at $x = 0$ and 1 and chaotic behaviour. Figure 4 shows a plot of the elliptic map $\phi_2^{(2)}(x, \alpha)$ for $\alpha_1 = 0.5$ and $\alpha_2 = 2.5$. In $\phi_2^{(2)}(x, \alpha_1)$ and $\phi_2^{(2)}(x, \alpha_2)$, x is limited to $\acute{x} \in [0, 0.28] \cup [0.72, 1]$ and $\ddot{x} \in [0.027, 0.973]$ respectively. Given $y_0 = 0.5$, $\acute{x}$ are the roots of $\phi_2^{(2)}(x, \alpha_1) = y_0$; similarly, $\ddot{x}_i$ are the roots of $\phi_2^{(2)}(x, \alpha_2) = y_0$.

3. Invariant measure

Invariant measure or SRB measure is supported on an attractor and describes the statistical of long-time behaviour of the orbits with respect to Lebesgue measure. For invariant measure of $\phi_N^{(i)}$ map ($i = 1, 2$) satisfying the Frobenius–Perron (FP) operator [18], we have

$$\mu(y) = \int_0^1 \delta(y - \phi_N^{(i)}(x, \alpha))\mu(x)\,\mathrm{d}x, \tag{3.1}$$

which is equivalent to

$$\mu(y) = \sum_{x \in \phi_N^{-1(i)}(y, \alpha)} \mu(x) \frac{\mathrm{d}x}{\mathrm{d}y}. \tag{3.2}$$

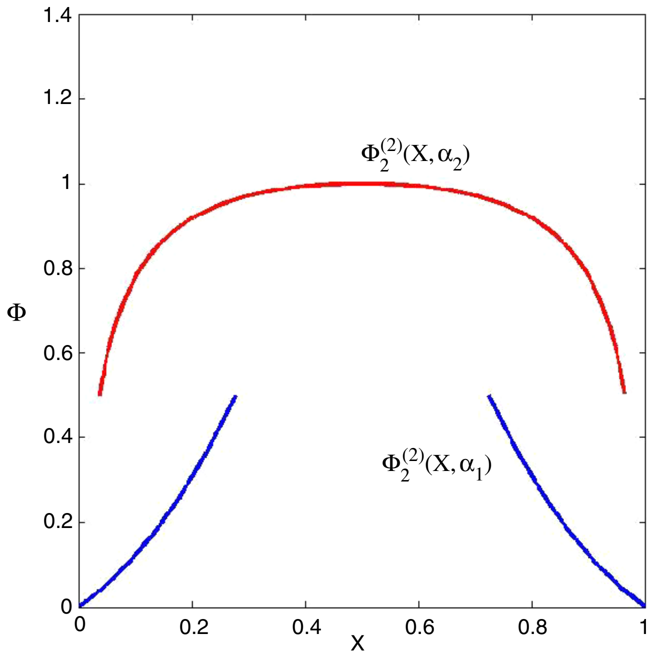

Figure 4. Plot of the elliptic map $\phi_2^{(2)}(x,\alpha)$ for $\alpha_1 = 0.5$ and $\alpha_2 = 2.5$.

For the chaotic part of the piecewise map, i.e. $y \in [0, y_0]$ for $\phi_2^{(1)}(x,\alpha)$ and $y \in [y_0, 1]$ for both $\phi_3^{(1,2)}(x,\alpha)$ and $\phi_2^{(2)}(x,\alpha)$, the invariant measure $\mu(x,\beta)$ is defined as

$$\frac{1}{\pi}\frac{\sqrt{\beta}}{\sqrt{x(1-x)}(\beta+(1-\beta)x)}, \tag{3.3}$$

where $\beta > 0$ is the invariant measure of the maps $\phi_N^{(i)}(x,\alpha)$ provided that we choose the parameter α in the following form:

$$\alpha = \frac{\sum_{k=0}^{[(N-1)/2]} C_{2k+1}^N \beta^{-k}}{\sum_{k=0}^{[N/2]} C_{2k}^N \beta^{-k}} \tag{3.4}$$

in $\phi_N^{(i)}(x,\alpha)$ maps for odd N, and

$$\alpha = \frac{\beta\sum_{k=0}^{[N/2]} C_{2k}^N \beta^{-k}}{\sum_{k=0}^{[(N-1)/2]} C_{2k+1}^N \beta^{-k}} \tag{3.5}$$

in $\phi_N^{(i)}(x,\alpha)$ maps for even N, where the symbol [ ] means the greatest integer part (the proof is presented in appendix A).

If an invariant measure is decomposed into parts that are invariant, the measure is called non-ergodic. There may be several invariant measures for a dynamic system. If there is a fixed point x^*, then a point distribution $\delta(x - x^*)$ in that point is an invariant measure even if the fixed point is unstable. Therefore, for the fixed-point part of the piecewise map, i.e. $y \in [y_0, 1]$ for $\phi_2^{(1)}(x,\alpha)$ and $y \in [0, y_0]$ for both maps $\phi_3^{(1,2)}(x,\alpha)$ and $\phi_2^{(2)}(x,\alpha)$, the average density measure $\mu(x,\beta)$ has the following asymptotic form of the delta function as α tends to

0 and 1, respectively:

$$\mu_{av}(x,\alpha) \xrightarrow{\alpha\to 0} \delta(x), \tag{3.6}$$

$$\mu_{av}(x,\alpha) \xrightarrow{\alpha\to 1} \delta(x-1), \tag{3.7}$$

where the first one corresponds to the invariant measure associated with the fixed point at $x = 0$ and the latter corresponds to the fixed point at $x = 1$.

Since, for small values of K, the parameter of the elliptic function, the elliptic chaotic maps of cn and sn types are topologically conjugate to the one-parameter families of trigonometric chaotic maps, we can obtain the invariant measure of these maps for small K [23]. As K vanishes, these maps are reduced to trigonometric chaotic maps.

4. KS entropy and Lyapunov exponents

KS entropy and Lyapunov characteristic exponents are two related ways of measuring 'disorder' in a dynamic system. A definition of them can be found in many textbooks [21]. To calculate KS entropy, we use the fact that it is equal to

$$h(\mu,\phi_N^{(i)}(x,\alpha)) = \int \mu(x)\,\mathrm{d}x \ln\left|\frac{\mathrm{d}}{\mathrm{d}x}\phi_N^{(i)}(x,\alpha)\right|, \tag{4.1}$$

which is also a statistical mechanical expression for the Lyapunov characteristic, i.e. the mean divergence rate of two nearby orbits. As shown in appendix B, the KS entropy of $\phi_N^{(i)}(x,\alpha)$ is given by the following expression:

$$h(\mu,\phi_N^{(i)}(x,\alpha)) = \ln\left(\frac{N(1+\beta+2\sqrt{\beta})^{N-1}}{\left(\sum_{k=0}^{[N/2]} C_{2k}^N\beta^k\right)\left(\sum_{k=0}^{[(N-1)/2]} C_{2k+1}^N\beta^k\right)}\right). \tag{4.2}$$

A useful numerical way to characterize chaotic phenomena in dynamic systems is by means of the Lyapunov exponents that describe the separation rate of systems, whose initial conditions differ by a small perturbation. Suppose that there is a small change $\delta x(0)$ in the initial state $x(0)$. At time t, this has changed to $\delta x(t)$ given by

$$\delta x(t) \approx \delta x(0)\left|\frac{\mathrm{d}\phi'}{\mathrm{d}x}(x(0))\right| = \delta x(0)|\phi'(x(t-1))\phi'(x(t-2))\ldots\phi'(x(0))|, \tag{4.3}$$

where we have used the chain rule to expand the derivative of ϕ. In the limit of infinitesimal perturbations $\delta x(0)$ and infinite time, we get an average exponential amplification, the Lyapunov exponent λ,

$$\lambda = \lim_{t\to\infty}\frac{1}{t}\ln\left|\frac{\delta x(t)}{\delta x(0)}\right| = \lim_{t\to\infty}\frac{1}{t}\ln\left|\frac{\mathrm{d}\phi'}{\mathrm{d}x}(x(0))\right| = \lim_{t\to\infty}\sum_{k=0}^{t-1}\ln|\phi'(x(k))|. \tag{4.4}$$

5. Ergodicity and non-ergodicity

A probabilistic dynamic system is characterized as ergodic or non-ergodic by its marginal probability distributions. If the distributions have, for example, infinite variances so that a process mean cannot be defined, then the system is non-ergodic. An ergodic system has 'convergent' qualities over time, variances are finite and a non-time-dependent process mean is clearly defined. Here, a brief description of the ergodic theory of chaos [19] is presented: let (Ω, F, μ) be a probability space, Ω is the sample space, i.e. the space of points, ω designating the elementary outcomes of an experiment and F is the σ-field (or σ-algebra) of events. An event is a set $A \subset \Omega$ that is of interest. The σ-field F is the ensemble of all events, i.e. $A \in F$. Also, μ designates a probability measure of F. A transformation T is ergodic, if it has the probability that, for almost every ω, the orbit $\{\omega, T\omega, T^2\omega, \ldots\}$ of ω is a sort of replica of Ω itself. Formally, we shall say that T is ergodic if each invariant set A, i.e. a set such that $T^{-1}(A) = A$, is trivial in the sense that it has a measure of either 0 or 1. $T^{-1}(A) = A \Rightarrow \mu(A) = 0$ or 1. Therefore, studies based on invariant measure analysis can be useful for confirming the non-ergodic behaviour of a map. In a non-ergodic system for a counter image set of $A \subset [0,1]$, we have

$$T^{-1}(A) = \{x \in [0, 1] \mid y = T(x), y \in A\} \tag{5.1}$$

and the map is non-ergodic if $0 < \mu(A) < 1$, i.e. the invariant measure that is not equal to 0 or 1 appears to be characteristic of non-ergodic behaviour. The transformation T is ergodic (or indecomposable or metrically transitive) if, in the Birkhoff theorem, for any integrable, real-value function f, the limit value $\hat{f}$ is constant and we have μ-almost everywhere:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k\omega)\,\mathrm{d}\mu(\omega) = \int_\Omega f(\omega)\,\mathrm{d}\mu(\omega). \tag{5.2}$$

In this case, the average value of $f(.)$, evaluated along the orbit $T^k\omega$, converges μ-almost everywhere to the mathematical expectation value or mean of $f(.)$, evaluated on the space Ω. In other words, for ergodic systems, the time average is equal to the space (or phase) average. One further consideration should be added at this point. The equality of KS entropy and the sum of all positive Lyapunov exponents, i.e.

$$h_{KS} = \sum_{\lambda_l > 0} \lambda_l, \tag{5.3}$$

indicates that, in a chaotic region, this map is ergodic as the Birkhoff ergodic theorem predicts [20]. In other words, when the KS entropy is smaller than the sum of positive LE, the map characterizes non-ergodic behaviour.

6. Results and discussion

In this section, we present the results of a numerical analysis for piecewise maps. Figures 5–8 show the variation of LE and KS entropy with the parameter α. A positive LE implies that two nearby trajectories diverge exponentially (at last locally). Negative LE indicates contraction along certain directions, and zero LE indicates that, along the relevant directions, there is neither expansion nor contraction.

In figure 5, the LE and the corresponding KS entropy is indicated for $\phi_2^{(1)}(x, \alpha_1)$ and $\phi_2^{(1)}(x, \alpha_2)$ by some points. A quantity of interest is that the KS entropy is smaller than the sum of the positive LE for piecewise maps. Because of this relation, it is clear that this map characterizes non-ergodic behaviour.

The invariant measure of this map is equal to 0.696, which is smaller than 1; therefore, this map behaves non-ergodically. The above analysis is presented for $\phi_2^{(2)}(x, \alpha)$, $\phi_3^{(1,2)}(x, \alpha)$ and the elliptic map $\phi_2^{(2)}(x, \alpha)$ (see figures 6–8). The invariant measure of these maps is equal to 0.6, 0.314 and 0.946, respectively, confirming the non-ergodic behaviour of the piecewise maps introduced.

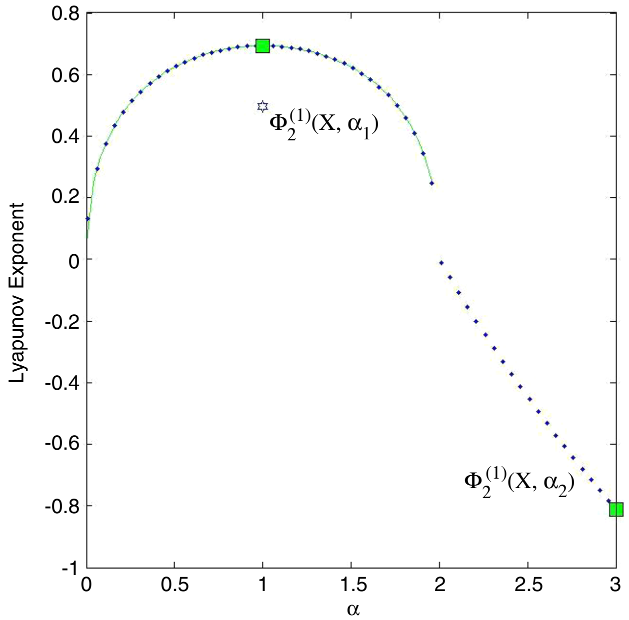

Figure 5. The variation of the Lyapunov exponent (dotted curve) and the KS entropy (solid curve) of $\phi_2^{(1)}(x, \alpha)$ with the parameter α. $\square$ and $\star$ show the values of the Lyapunov exponent and the KS entropy, respectively, for $\phi_2^{(1)}(x, \alpha_1)$ and $\phi_2^{(1)}(x, \alpha_2)$.

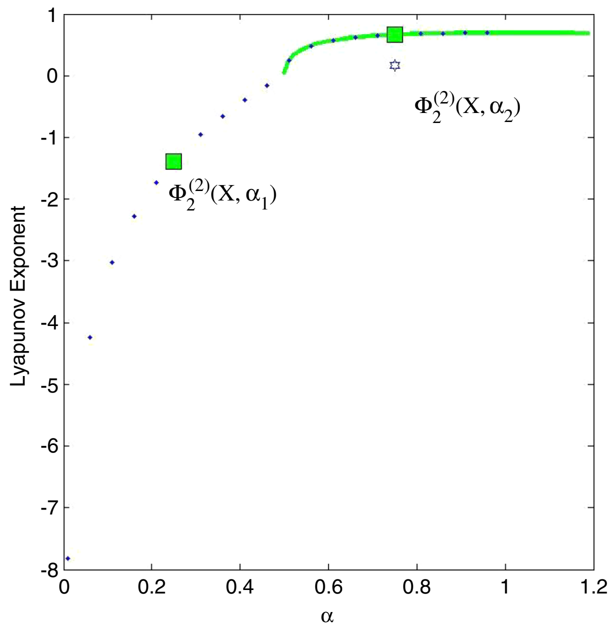

Figure 6. The variation of the Lyapunov exponent (dotted curve) and the KS entropy (solid curve) of $\phi_2^{(2)}(x, \alpha)$ with the parameter α. $\square$ and $\star$ show the values of the Lyapunov exponent and the KS entropy, respectively, for $\phi_2^{(2)}(x, \alpha_1)$ and $\phi_2^{(2)}(x, \alpha_2)$.

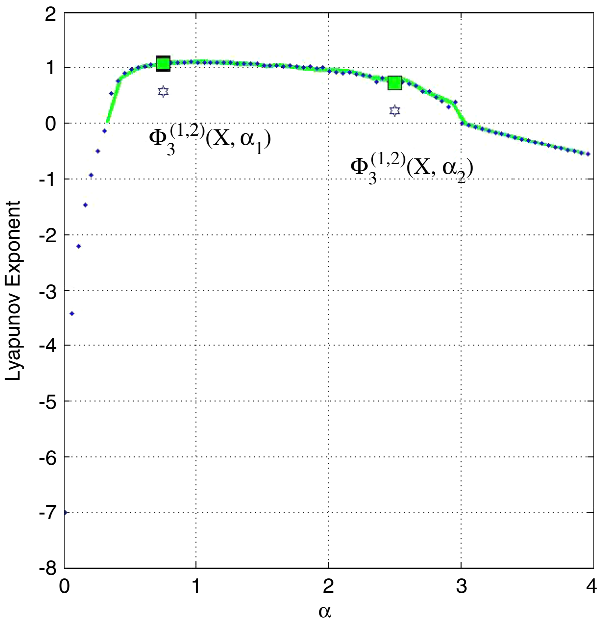

Figure 7. The variation of the Lyapunov exponent (dotted curve) and the KS entropy (solid curve) of $\phi_3^{(1,2)}(x, \alpha)$ with the parameter α. $\square$ and $\star$ show the values of the Lyapunov exponent and the KS entropy, respectively, for $\phi_3^{(1,2)}(x, \alpha_1)$ and $\phi_3^{(1,2)}(x, \alpha_2)$.

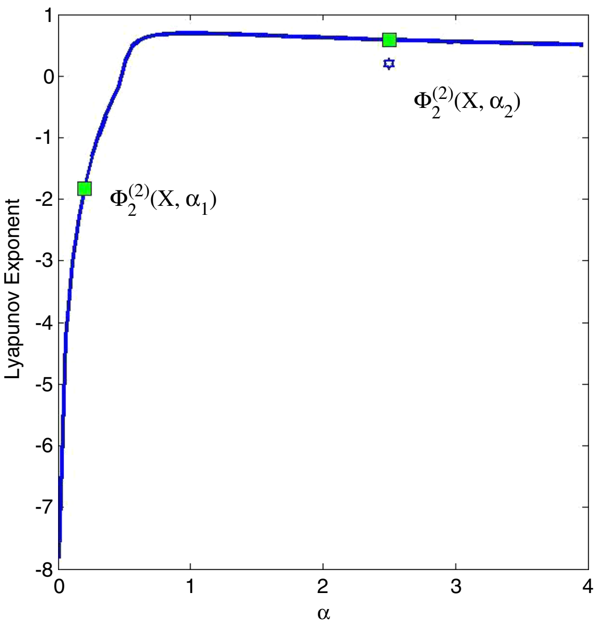

Figure 8. The variation of the Lyapunov exponent (solid curve) of the elliptic map $\phi_2^{(2)}(x, \alpha)$ with the parameter α for $K = 0$. $\square$ and $\star$ show the values of the Lyapunov exponent and the KS entropy, respectively, for $\phi_2^{(2)}(x, \alpha_1)$ and $\phi_2^{(2)}(x, \alpha_2)$.

7. Conclusion

Recent attempts at introducing the hierarchy of chaotic maps with their invariant measure [16, 17, 22–24] allow us to advance, in answering to a question, how to define non-ergodic maps and what are the conditions for non-ergodicity in these types of systems.

In this paper, we introduce the piecewise maps with their invariant measure. Our numerical calculations show that, for introduced piecewise maps, values of the KS entropy are smaller compared with the sum of positive LE; therefore, these maps behave non-ergodically. Together with the non-ergodic behaviour, we also find that the invariant measure is different from 0 or 1.

Appendix A

Similar to the calculation of the invariant measure in our previous papers [16, 17, 22–24], we present here that for the piecewise chaotic map. To prove that the measure (3.3) satisfied equation (3.2), we consider the conjugate map

$$\tilde{\phi}_N^{(1)}(x,\alpha) = \frac{1}{\alpha^2}\tan^2(N\arctan\sqrt{x}) \tag{A.1}$$

with the measure $\tilde{\mu}_{\tilde{\phi}_N}$ related to the measure μ_{ϕ_N} by the following relation:

$$\tilde{\mu}_{\tilde{\phi}_N}(x) = \frac{1}{(1+x)^2}\mu_{\phi_N}\left(\frac{1}{1+x}\right).$$

Denoting $\tilde{\phi}_N(x,\alpha)$ on the left-hand side of (A.1) by y and inverting it, we get

$$x_k = \tan^2\left(\frac{1}{N}\arctan\sqrt{y\alpha^2} + \frac{k\pi}{N}\right), \quad k = 1,\ldots,N. \tag{A.2}$$

Then, taking the derivative of x_k with respect to y, we obtain

$$\left|\frac{\mathrm{d}x_k}{\mathrm{d}y}\right| = \frac{\alpha}{N}\sqrt{x_k}(1+x_k)\frac{1}{\sqrt{y}(1+\alpha^2 y)}. \tag{A.3}$$

Substituting the above result into equation (3.2), we get

$$\tilde{\mu}_{\tilde{\phi}_N}(y)\sqrt{y}(1+\alpha^2 y) = \frac{\alpha}{N}\sum_k \sqrt{x_k}(1+x_k)\tilde{\mu}_{\tilde{\phi}_N}(x_k). \tag{A.4}$$

Now, by considering the following ansatz for the invariant measure $\tilde{\mu}_{\tilde{\phi}_N}(y)$:

$$\tilde{\mu}_{\tilde{\phi}_N}(y) = \frac{\sqrt{\beta}}{\sqrt{y}(1+\beta y)}, \tag{A.5}$$

the above equation reduces to

$$\frac{1+\alpha^2 y}{1+\beta y} = \frac{\alpha}{N}\sum_{k=1}^{N}\left(\frac{1+x_k}{1+\beta x_k}\right),$$

which can be written as

$$\frac{1+\alpha^2 y}{1+\beta y} = \frac{\alpha}{\beta} + \left(\frac{\beta-1}{\beta^2}\right)\frac{\partial}{\partial\beta^{-1}}\left(\ln\left(\prod_{k=1}^{N}(\beta^{-1}+x_k)\right)\right). \tag{A.6}$$

To evaluate the second term on the right-hand side of the above formulae, we can write the equation in the following form:

$$
\begin{aligned}
0 &= \alpha^2 y \cos^2(N \arctan \sqrt{x}) - \sin^2(N \arctan \sqrt{x}) \\
&= \frac{(-1)^N}{(1+x)^N}\left(\alpha^2 y \left(\sum_{k=0}^{[N/2]} C_{2k}^N (-1)^N x^k\right)^2 - x\left(\sum_{k=0}^{[(N-1)/2]} C_{2k+1}^N (-1)^N x^k\right)^2\right) \\
&= \frac{\text{constant}}{(1+x)^N}\prod_{k=1}^{N}(x - x_k),
\end{aligned}
$$

where x_k are the roots of equation (A.1) and they are given by the formula (A.2). Therefore, we have

$$
\begin{aligned}
\frac{\partial}{\partial \beta^{-1}} \ln\left(\prod_{k=1}^{N}(\beta^{-1} + x_k)\right) &= \frac{\partial}{\partial \beta^{-1}} \ln((1-\beta^{-1})^N(\alpha^2 y \cos^2(N \arctan \sqrt{-\beta^{-1}}) \\
&\quad - \sin^2(N \arctan \sqrt{-\beta^{-1}})) = -\frac{N\beta}{\beta - 1} + \frac{\beta N(1+\alpha^2 y)A(1/\beta)}{(A(1/\beta))^2\beta^2 y + (B(1/\beta))^2}, \quad \text{(A.7)}
\end{aligned}
$$

whereby the polynomials $A(x)$ and $B(x)$ are defined by

$$
A(x) = \sum_{k=0}^{[N/2]} C_{2k}^N x^k, \quad \text{(A.8)}
$$

$$
B(x) = \sum_{k=0}^{[(N-1)/2]} C_{2k+1}^N x^k. \quad \text{(A.9)}
$$

In deriving the above formula, we have used the following identities:

$$
\cos(N \arctan \sqrt{x}) = \frac{A(-x)}{(1+x)^{N/2}}, \qquad \sin(N \arctan \sqrt{x}) = \sqrt{x}\frac{B(-x)}{(1+x)^{N/2}}. \quad \text{(A.10)}
$$

Inserting the result (A.7) into (A.6), we get

$$
\frac{1+\alpha^2 y}{1+\beta y} = \frac{1+\alpha^2 y}{B(1/\beta)/A(1/\beta) + \beta(\alpha A(1/\beta)/B(1/\beta))y}. \quad \text{(A.11)}
$$

Hence, to get the final result, we have to choose the parameter α as

$$
\alpha = \frac{B(1/\beta)}{A(1/\beta)}. \quad \text{(A.12)}
$$

Appendix B

The KS entropy of one-parameter families of chaotic map is given by equation (4.1) i.e.

$$
h(\mu, \phi(x,\alpha)) = \int \mu(x)\,\mathrm{d}x \ln\left|\frac{\mathrm{d}}{\mathrm{d}x}\phi(x,\alpha)\right|,
$$

where

$$\varphi(x,\alpha) = y = \frac{1}{\alpha^2}(\tan^2(N \arctan\sqrt{x})).$$

Therefore, to calculate $h(\mu, \varphi(x,\alpha))$, we have

$$h(\mu,\varphi(x,\alpha)) = \int_0^\infty \tilde{\mu}(x)\,\mathrm{d}x \ln\left(\left|\frac{N}{\alpha^2}\frac{1}{\sqrt{x}(1+x)}\frac{\sin N(\arctan\sqrt{x})}{\cos^3 N(\arctan\sqrt{x})}\right|\right).$$

Using relation (A.8), we get

$$h(\mu,\varphi(x,\alpha)) = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{\beta}\,\mathrm{d}x}{\sqrt{x}(1+\beta x)}\ln\left(\left|\frac{N}{\alpha^2}\frac{(1+x)^{N-1}B(-x)}{(A(-x))^3}\right|\right). \tag{B.1}$$

We see that polynomials appearing in the numerator (denominator) of the integrand appearing on the right-hand side of equation (B.1) have $\frac{1}{2}[N-1]$ ($\frac{1}{2}[N]$) simple roots, denoted by x_k^B, $k = 1,\ldots,\frac{1}{2}[N-1]$ (x_k^A, $k = 1,\ldots,\frac{1}{2}[N]$) in the interval $[0,\infty)$. Hence, we can write the above formula in the following form:

$$h(\mu,\varphi(x,\alpha)) = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{\beta}\,\mathrm{d}x}{\sqrt{x}(1+\beta x)}\ln\left(\frac{N}{\alpha^2}\frac{(1+x)^{N-1}\prod_{k=1}^{[(N-1)/2]}|x-x_k^B|}{\prod_{k=1}^{[N/2]}|x-x_k^A|}\right). \tag{B.2}$$

Now, making the following change of the variable $\sqrt{\beta}x = \tan\theta$ and taking into account that the degrees of numerator and denominator are equal for both even and odd values on N we get

$$\begin{aligned} h(\mu,\varphi(x,\alpha)) = \frac{1}{\pi}\int_0^\infty \mathrm{d}\theta \Bigg\{ &\ln\left(\frac{N}{\alpha^2}\right) + (N-1)\ln|\beta+1+(\beta-1)\cos\theta| \\ &+ \sum_{k=1}^{[(N-1)/2]} \ln|1-x_k^B\beta+(1+x_k^B\beta)\cos\theta| \\ &- 3\sum_{k=1}^{[N/2]} \ln|1-x_k^A\beta+(1+x_k^A\beta)\cos\theta| \Bigg\} \end{aligned} \tag{B.3}$$

using the following integrals:

$$\frac{1}{\pi}\int_0^\pi \ln|a+b\cos\theta| = \begin{cases} \ln\left|\dfrac{a+\sqrt{a^2-b^2}}{2}\right|, & |a|>|b|, \\ \ln\left|\dfrac{b}{2}\right|, & |a|\leqslant|b|. \end{cases} \tag{B.4}$$

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